Vancouver School of Economics, University of British Columbia

Notes for tutorial 1

I. The CRRA and the CARA utility functions.

Consider an infinitely lived consumer whose preferences are defined over the consumption of single good, c_t. Household's objective is to maximize lifetime utility:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(c_t)$ is the Constant Relative Risk Aversion (CRRA) utility function given by $u(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$, where $\sigma \geq 0$ is a parameter.

1. Use L'Hospital's rule to show that $\lim_{\sigma \to 1} u(c_t) = \ln(c_t)$, so that log utility is a special case of CRRA utility function.
   A: The (L'Hospital's Rule) states:
   If $f, g : (a, b) \Rightarrow \mathbb{R}$ are differentiable and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$ exists and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then
   $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = A.$$
   Let $1 - \sigma = \alpha$. Then
   $$\lim_{\alpha \to 0} \frac{c_t^{\alpha} - 1}{\alpha} = \lim_{\alpha \to 0} \frac{\ln c_t * c_t^{\alpha}}{1} = \ln c_t$$
2. Show that the CRRA utility function with $\sigma > 0$ is strictly increasing, strictly concave and satisfies the Inada conditions. Recall that Inada conditions require
   $$\lim_{c \to 0} u'(c) = +\infty$$
   $$\lim_{c \to +\infty} u'(c) = 0.$$
   A: $u'(c) = c_t^{-\sigma} > 0, u''(c_t) = -\sigma c_t^{-\sigma-1} < 0.$
3. Define $-\frac{u''(c_t)c_t}{u'(c_t)}$ to be the (Arrow-Pratt) coefficient of relative risk aversion. It indicates household's attitude towards risk. Show that the CRRA utility function has a constant Arrow-Pratt coefficient of relative risk aversion equal to σ.
   A: $-\frac{u''(c_t)c_t}{u'(c_t)} = -\frac{-\sigma c_t^{-\sigma-1} c}{c_t^{-\sigma}} = \sigma$
4. Show that the CRRA utility function has a constant intertemporal elasticity of substitution equal to $\frac{1}{\sigma}$.
   A: see class notes

5. Define the marginal rate of substitution between consumption at any two dates t and $t+s$ as

$$MRS(c_{t+s}, c_t) = \frac{\partial u(c_{t+s})/\partial c_{t+s}}{\partial u(c_t)/\partial c_t}.$$

The function u is said to be homothetic if $MRS(c_{t+s}, c_t) = MRS(\gamma c_{t+s}, \gamma c_t)$ for all $\gamma > 0$. Show that if $u(c_t)$ is of CRRA form, then $u(c)$ is homothetic.

A:

$$MRS(c_{t+s}, c_t) = \frac{c_{t+s}^{-\sigma}}{c_t^{-\sigma}} = \frac{(\gamma c_{t+s})^{-\sigma}}{(\gamma c_t)^{-\sigma}} = MRS(\gamma c_{t+s}, \gamma c_t).$$

6. Another period utility function that we will sometimes use is the Constant Absolute Risk Aversion (CARA) utility function given by $u(c_t) = 1 - \exp\{-\alpha c_t\}$, where α is a parameter.

   (a) Define the (Arrow-Pratt) coefficient of absolute risk aversion as $-\frac{u''(c_t)}{u'(c_t)}$. Show that the CARA utility function has a constant absolute risk aversion coefficient, but an increasing relative risk aversion coefficient.

   A:

$$\begin{aligned} u'(c) &= \alpha \exp\{-\alpha c_t\} \\ u''(c) &= -\alpha^2 \exp\{-\alpha c_t\} \end{aligned}$$

   Then the coefficient of absolute risk aversion is $-\frac{u''(c_t)}{u'(c_t)} = -\frac{-\alpha^2 \exp\{-\alpha c_t\}}{\alpha \exp\{-\alpha c_t\}} = \alpha$. While the coefficient of relative risk aversion is $-\frac{u''(c_t)c_t}{u'(c_t)} = -\frac{-\alpha^2 \exp\{-\alpha c_t\}c_t}{\alpha \exp\{-\alpha c_t\}} = \alpha c_t$.

   (b) Is CARA utility function homothetic? Show your answer formally.

   A:

$$\begin{aligned} MRS(c_{t+s}, c_t) &= \frac{\exp\{-\alpha c_{t+s}\}}{\exp\{-\alpha c_t\}} \\ MRS(\gamma c_{t+s}, \gamma c_t) &= \frac{\exp\{-\gamma \alpha c_{t+s}\}}{\exp\{-\gamma \alpha c_t\}} = [MRS(c_{t+s}, c_t)]^{\gamma}. \end{aligned}$$

   Thus, utility function of CARA class are not homothetic.

   (c) Does CARA utility function satisfy the Inada conditions? Show your answer formally.

   A:

$$\begin{aligned} \lim_{c \to 0} u'(c) &= \lim_{c \to 0} \alpha \exp\{-\alpha c_t\} = \alpha \\ \lim_{c \to +\infty} u'(c) &= \lim_{c \to 0} \alpha \exp\{-\alpha c_t\} = 0. \end{aligned}$$

II. Constrained optimization: Lagrangeans and Kuhn-Tucker Conditions

- Equality constraints (Lagrangians)
- Inequality constraints (Kuhn-Tuckers)
- Necessary and sufficient conditions
- Examples